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# Lagrangian relative equilibria for a gyrostat in the three-body problem: bifurcations and stability

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**Abstract**
In this paper we consider the non-canonical Hamiltonian dynamics of a gyrostat in the frame of the three-body problem. Using geometric/mechanic methods we study the approximate dynamics of the truncated Legendre series representation of the potential of an arbitrary order. Working in the reduced problem, we study the existence of relative equilibria that we refer to as Lagrange type following the analogy with the standard techniques. We provide necessary and sufficient conditions for the linear stability of Lagrangian relative equilibria if the gyrostat morphology form is close to a sphere. Thus, we generalize the classical results on equilibria of the three-body problem and many results on them obtained by the classic approach for the case of rigid bodies.

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(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

A new interest in the study of configurations of relative equilibria by the use of differential geometry methods instead of more classical ones has appeared in recent years. See for instance, Wang *et al* [8] who treat the problem of a rigid body in a central Newtonian field or Maciejewski [5] for the problem of two rigid bodies under mutual Newtonian attraction.

A *gyrostat* is a mechanical system $S$ composed by a rigid body $S'$ and other bodies $S''$ deformable or rigid, connected in such a way that their relative motion with respect to its rigid part does not change the distribution of masses of the total system (see Leimanis [4]). Results

  

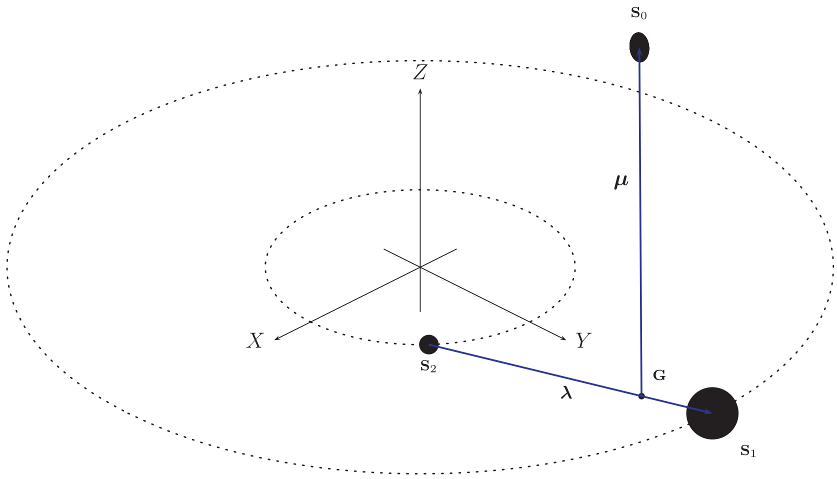


**Figure 1.** Gyrostat in the three-body problem.

of papers [5] and [8] are generalized in Mondéjar *et al* [3] to the case of two gyrostats under mutual Newtonian attraction.

Concerning the problem of the motion of three rigid bodies, Vidiakin [6] and Duboshin [1] prove the existence of Euler's and Lagrange's configurations of equilibria when the physical morphology of the bodies has symmetries (for a more up-to-date reference see [9]). See Guirao *et al* [2] for a complete study of the Eulerian relative equilibria, see figure 1.

Vera [7] and Vera *et al* [9] have studied the non-canonical Hamiltonian dynamics of $n + 1$ bodies in Newtonian attraction, $n$ of them being rigid bodies with a spherical distribution of masses or material points and the other one a triaxial gyrostat. Working on the reduced problem, global considerations on the conditions for relative equilibria are performed.

In this paper, we consider the problem of analyzing the non-canonical Hamiltonian dynamics of two bodies in Newtonian attraction from a qualitative point of view. Thus, we shall describe the approximate dynamics appearing when we take the truncated Legendre series representation of the potential function at an arbitrary order.

We provide global conditions on the existence of relative equilibria in the case where $S_1$ and $S_2$ are spherical or punctual bodies and $S_0$ is a gyrostat. Following the analogy with the classical results we shall refer to such equilibria as *Lagrange* type. Necessary and sufficient conditions for the existence of such equilibria are stated and their explicit expressions are presented. It allows us to study their stability. We develop a complete study of the linear stability of Lagrangian relative equilibria when the gyrostat morphology form is close to a sphere.

As a consequence of this geometric/mechanic study we obtain and generalize some results previously stated using classical methods in previous works. On the other hand, new results not obtained with standard techniques are presented.

The methods introduced in this work can be used in similar problems. A natural extension of this work, which we state as a problem for the future, is to study the nonlinear stability of the Lagrangian relative equilibria obtained in this paper.

## 2. Equations of motion

Let $S_0$ be a gyrostat of mass $m_0$ and $S_1$, $S_2$ be two spherical rigid bodies of masses $m_1$ and $m_2$, respectively. We use the following notations:

$$M_2 = m_1 + m_2, \qquad M_1 = m_1 + m_2 + m_0, \qquad g_1 = \frac{m_1 m_2}{M_2}, \qquad g_2 = \frac{m_0 M_2}{M_1}.$$

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, $\mathbf{u} \cdot \mathbf{v}$ is the dot product, $|\mathbf{u}|$ is the Euclidean norm of the vector $\mathbf{u}$ and $\mathbf{u} \times \mathbf{v}$ is the cross product. $\mathbf{I}_{\mathbb{R}^3}$ is the identity matrix and $\mathbf{0}$ is the zero matrix of order 3. We consider $\mathbb{I} = \mathrm{diag}(A, A, C)$ the diagonal tensor of inertia of the gyrostat and let $\mathbf{z} = (\mathbf{\Pi}, \lambda, \mathbf{p}_\lambda, \mu, \mathbf{p}_\mu) \in \mathbb{R}^{15}$ be a generic element of the twice reduced problem obtained using the symmetries of the system. $\mathbf{\Pi} = \mathbb{I}\mathbf{\Omega} + \mathbf{l}_r$ is the total rotational angular momentum vector of the gyrostat in the body frame, which is attached to its rigid part and whose axes have the direction of the principal axes of inertia of $S_0$ and $\mathbf{l}_r = (0, 0, l)$ is the constant gyrostatic momentum. The elements $\lambda$, $\mu$, $\mathbf{p}_\lambda$ and $\mathbf{p}_\mu$ are respectively the barycentric coordinates and the linear momenta expressed in the body frame $\mathfrak{J}$.

The twice reduced Hamiltonian of the system, see [9] for more details, has the following expression:

$$\mathcal{H}(\mathbf{z}) = \frac{|\mathbf{p}_\lambda|^2}{2g_1} + \frac{|\mathbf{p}_\mu|^2}{2g_2} + \frac{1}{2}\Pi\, \mathbb{I}^{-1}\Pi - \mathbf{l}_r \cdot \mathbb{I}^{-1}\Pi + \mathcal{V}(\lambda, \mu). \tag{1}$$

Let $\mathbf{M} = \mathbb{R}^{15}$, and we consider the manifold $(\mathbf{M}, \{\ ,\ \}, \mathcal{H})$, with Poisson brackets $\{\ ,\ \}$, defined by using the the Poisson tensor

$$\mathbf{B}(\mathbf{z}) = \begin{pmatrix} \widehat{\mathbf{\Pi}} & \widehat{\lambda} & \widehat{\mathbf{p}_\lambda} & \widehat{\mu} & \widehat{\mathbf{p}_\mu} \\ \widehat{\lambda} & \mathbf{0} & \mathbf{I}_{\mathbb{R}^3} & \mathbf{0} & \mathbf{0} \\ \widehat{\mathbf{p}_\lambda} & -\mathbf{I}_{\mathbb{R}^3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \widehat{\mu} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{\mathbb{R}^3} \\ \widehat{\mathbf{p}_\mu} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathbb{R}^3} & \mathbf{0} \end{pmatrix}. \tag{2}$$

In $\mathbf{B}(\mathbf{z})$, $\widehat{\mathbf{v}}$ is considered to be the image of the vector $\mathbf{v} \in \mathbb{R}^3$ by the standard isomorphism between the Lie Algebras $\mathbb{R}^3$ and $\mathfrak{so}(3)$, i.e.

$$\widehat{\mathbf{v}} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}.$$

The equations of the motion are given by the following expression:

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \{\mathbf{z}, \mathcal{H}(\mathbf{z})\} = \mathbf{B}(\mathbf{z})\nabla_{\mathbf{z}}\mathcal{H}(\mathbf{z})$$

where $\nabla_{\mathbf{u}}\mathcal{V}$ is the gradient of $\mathcal{V}$ with respect to an arbitrary vector $\mathbf{u}$.

Developing $\{\mathbf{z}, \mathcal{H}(\mathbf{z})\}$, we obtain the following group of vectorial equations of the motion:

$$\begin{aligned} &\frac{\mathrm{d}\mathbf{\Pi}}{\mathrm{d}t} = \mathbf{\Pi} \times \mathbf{\Omega} + \lambda \times \nabla_\lambda \mathcal{V} + \mu \times \nabla_\mu \mathcal{V}, \\ &\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{\mathbf{p}_\lambda}{g_1} + \lambda \times \mathbf{\Omega}, \qquad \frac{\mathrm{d}\mathbf{p}_\lambda}{\mathrm{d}t} = \mathbf{p}_\lambda \times \mathbf{\Omega} - \nabla_\lambda \mathcal{V}, \\ &\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{\mathbf{p}_\mu}{g_2} + \mu \times \mathbf{\Omega}, \qquad \frac{\mathrm{d}\mathbf{p}_\mu}{\mathrm{d}t} = \mathbf{p}_\mu \times \mathbf{\Omega} - \nabla_\mu \mathcal{V}. \end{aligned} \tag{3}$$

We denote by $\mathbf{z}_e = (\mathbf{\Pi}_e, \lambda^e, \mathbf{p}_\lambda^e, \mu^e, \mathbf{p}_\mu^e)$ a generic relative equilibrium of

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \{\mathbf{z}, \mathcal{H}(\mathbf{z})\} = \mathbf{B}(\mathbf{z})\nabla_\mathbf{z}\mathcal{H}(\mathbf{z}).$$

The potential function $\mathcal{V}(\lambda, \mu)$ is given by the expression

$$-\left(\frac{Gm_1m_2}{|\lambda|} + Gm_1\int_{S_0}\frac{\mathrm{d}m(\mathbf{Q})}{\left|\mathbf{Q} + \mu + \frac{m_2}{M_2}\lambda\right|} + Gm_2\int_{S_0}\frac{\mathrm{d}m(\mathbf{Q})}{\left|\mathbf{Q} + \mu - \frac{m_1}{M_2}\lambda\right|}\right). \quad (4)$$

## 3. Approximate Hamiltonian dynamics

To simplify the problem we assume that the gyrostat $S_0$ is symmetrical around the third axis of inertia $Oz$ of the body frame $\mathfrak{J}$ and with respect to the plane $Oxy$ of the same frame. If the mutual distances are longer than the individual dimensions of the bodies, then we can develop the potential using a convergent series of high speed. Under these hypotheses, we will be able to carry out a study of certain relative equilibria in different approximate dynamics.

Applying the Legendre development of the potential, we have that $\mathcal{V}(\lambda, \mu)$ has the form

$$-\left(\frac{Gm_1m_2}{|\lambda|} + Gm_1\sum_{i=0}^{\infty}\frac{A_{2i}}{\left|\mu + \frac{m_2}{M_2}\lambda\right|^{2i+1}} + Gm_2\sum_{i=0}^{\infty}\frac{A_{2i}}{\left|\mu - \frac{m_1}{M_2}\lambda\right|^{2i+1}}\right)$$

where $A_0 = m_0$, $A_2 = (C - A)/2$ and $A_{2i}$ are certain coefficients related to the geometry of the gyrostat, see [9] for details.

**Definition 1.** *The approximate potential of order k $\mathcal{V}^{(k)}(\lambda, \mu)$ is defined as the following expression:*

$$-\left(\frac{Gm_1m_2}{|\lambda|} + Gm_1\sum_{i=0}^{k}\frac{A_{2i}}{\left|\mu + \frac{m_2}{M_2}\lambda\right|^{2i+1}} + Gm_2\sum_{i=0}^{k}\frac{A_{2i}}{\left|\mu - \frac{m_1}{M_2}\lambda\right|^{2i+1}}\right).$$

**Definition 2.** *Let $\mathbf{M} = \mathbb{R}^{15}$ and let the manifold $(M, \{\ ,\ \}, \mathcal{H}^k)$, with Poisson brackets $\{\ ,\ \}$, be defined by using the Poisson tensor*

$$\mathbf{B}(\mathbf{z}) = \begin{pmatrix} \widehat{\mathbf{\Pi}} & \widehat{\lambda} & \widehat{\mathbf{p}_\lambda} & \widehat{\mu} & \widehat{\mathbf{p}_\mu} \\ \widehat{\lambda_1} & \mathbf{0} & \mathbf{I}_{\mathbb{R}^3} & \mathbf{0} & \mathbf{0} \\ \widehat{\mathbf{p}_\lambda} & -\mathbf{I}_{\mathbb{R}^3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \widehat{\mu} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{\mathbb{R}^3} \\ \widehat{\mathbf{p}_\mu} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathbb{R}^3} & \mathbf{0} \end{pmatrix}.$$

*We define the approximate dynamics of order k to be the differential equations of motion given by the following expression:*

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \{\mathbf{z}, \mathcal{H}^k(\mathbf{z})\} = \mathbf{B}(\mathbf{z})\nabla_\mathbf{z}\mathcal{H}^k(\mathbf{z})$$

*where*

$$\mathcal{H}^k(\mathbf{z}) = \frac{|\mathbf{p}_\lambda|^2}{2g_1} + \frac{|\mathbf{p}_\mu|^2}{2g_2} + \frac{1}{2}\Pi\mathbb{I}^{-1}\Pi - \mathbf{l}_r \cdot \mathbb{I}^{-1}\Pi + \mathcal{V}^{(k)}(\lambda, \mu).$$

In this setting we have the following result.

**Theorem 3.** *In the approximate dynamics of order 0, $|\mathbf{\Pi}|^2$ is an integral of motion and also when the gyrostat is of revolution $\pi_3$ is another integral of motion for all approximate dynamics.*

**Proof.** The proof is the consequence of two facts. On the one hand, by calculation it is easy to verify that

$$\nabla_{\mathbf{z}}(|\mathbf{\Pi}|^2))\mathbf{B}(\mathbf{z})\nabla_{\mathbf{z}}\mathcal{H}^0(\mathbf{z}) = 0$$

and on the other hand, we obtain in a similar way, when the gyrostat is of revolution,

$$\nabla_{\mathbf{z}}(\pi_3)\mathbf{B}(\mathbf{z})\nabla_{\mathbf{z}}\mathcal{H}^k(\mathbf{z}) = 0$$

where $\pi_3$ is the third component of the rotational angular momentum of the gyrostat. □

## 4. Relative equilibria

The relative equilibria are the equilibria of the twice reduced problem whose Hamiltonian function is obtained in [9] for the case $n = 2$. If we denote by $\mathbf{z}_e = \left(\mathbf{\Pi}_e, \lambda^e, \mathbf{p}^e_\lambda, \mu^e, \mathbf{p}^e_\mu\right)$ a generic relative equilibrium of an approximate dynamics of order $k$, then this verifies the equations

$$\begin{aligned}
&\mathbf{\Pi}_e \times \mathbf{\Omega}_e + \lambda^e \times (\nabla_\lambda \mathcal{V}^{(k)})_e + \mu^e \times (\nabla_\mu \mathcal{V}^{(k)})_e = \mathbf{0}, \\
&\frac{\mathbf{p}^e_\lambda}{g_1} + \lambda^e \times \mathbf{\Omega}_e = \mathbf{0}, \qquad \mathbf{p}^e_\lambda \times \mathbf{\Omega}_e = (\nabla_\lambda \mathcal{V}^{(k)})_e, \\
&\frac{\mathbf{p}^e_\mu}{g_2} + \mu^e \times \mathbf{\Omega}_e = \mathbf{0}, \qquad \mathbf{p}^e_\mu \times \mathbf{\Omega}_e = (\nabla_\mu \mathcal{V}^{(k)})_e
\end{aligned} \tag{5}$$

where $(\nabla_\lambda \mathcal{V}^{(k)})_e$ and $(\nabla_\mu \mathcal{V}^{(k)})_e$ respectively are the values of $\nabla_\lambda \mathcal{V}^{(k)}$ and $\nabla_\mu \mathcal{V}^{(k)}$ in $\mathbf{z}_e$.

The following result which is provided in [9] will play a key role in this work because it will be used to obtain necessary conditions for the existence of relative equilibria in approximate dynamics.

**Lemma 4.** *If $\mathbf{z}_e = \left(\mathbf{\Pi}_e, \lambda^e, \mathbf{p}^e_\lambda, \mu^e, \mathbf{p}^e_\mu\right)$ is a relative equilibrium of an approximate dynamics of order $k$, the following relationships are verified:*

$$|\mathbf{\Omega}_e|^2|\lambda^e|^2 - (\lambda^e \cdot \mathbf{\Omega}_e)^2 = \frac{1}{g_1}(\lambda^e \cdot (\nabla_\lambda \mathcal{V}^{(k)})_e)$$

$$|\mathbf{\Omega}_e|^2|\mu^e|^2 - (\mu^e \cdot \mathbf{\Omega}_e)^2 = \frac{1}{g_2}(\mu^e \cdot (\nabla_\mu \mathcal{V}^{(k)})_e).$$

We will study the relative equilibria in the approximate dynamics for which their vectors $\mathbf{\Omega}_e$, $\lambda^e$, $\mu^e$ satisfy some special geometric properties.

**Definition 5.** *A point $\mathbf{z}_e$ will be called the Lagrangian relative equilibrium in an approximate dynamics of order k, if $\lambda^e$, $\mu^e$ are not proportional vectors and $\mathbf{\Omega}_e$ is perpendicular to the plane that $\lambda^e$ and $\mu^e$ generate.*

In this setting we have the following result.

**Proposition 6.** *In a Lagrangian relative equilibrium for any approximate dynamics of arbitrary order, moments are not exercised on the gyrostat.*

**Proof.** The proof follows from the equations of motion and the potential relations in the equilibrium. □

In the next section we endeavor to obtain necessary and sufficient conditions for the existence of Lagrangian relative equilibria.

## 5. Lagrangian relative equilibria

### *5.1. Necessary condition for the existence*

**Proposition 7.** *Let* $\mathbf{z}_e = \left(\mathbf{\Pi}_e, \lambda^e, \mathbf{p}_\lambda^e, \mu^e, \mathbf{p}_\mu^e\right)$ *be Lagrangian relative equilibria. Then we have*

$$g_2(\widetilde{A}_{11})_e = g_1(\widetilde{A}_{22})_e$$
$$(\widetilde{A}_{12})_e = 0$$

*with*

$$|\mathbf{\Omega}_e|^2 = \frac{(\widetilde{A}_{11})_e}{g_1} = \frac{(\widetilde{A}_{22})_e}{g_2}.$$

**Proof.** If $\mathbf{z}_e = \left(\mathbf{\Pi}_e, \lambda^e, \mathbf{p}_\lambda^e, \mu^e, \mathbf{p}_\mu^e\right)$ is Lagrangian relative equilibria, in an approximate dynamics of order $k$, the following identities are verified:

$$\lambda^e \times (\nabla_\lambda \mathcal{V}^{(k)})_e = \mathbf{0}, \qquad g_1|\mathbf{\Omega}_e|^2(\lambda^e \times \mu^e) = (\nabla_\lambda \mathcal{V}^{(k)})_e \times \mu^e,$$
$$\mu^e \times (\nabla_\mu \mathcal{V}^{(k)})_e = \mathbf{0}, \qquad g_2|\mathbf{\Omega}_e|^2(\lambda^e \times \mu^e) = \lambda^e \times (\nabla_\mu \mathcal{V}^{(k)})_e.$$

In the relative equilibria, from equation (A.2) of the appendix, we deduce

$$(\widetilde{A}_{12})_e(\lambda^e \times \mu^e) = \mathbf{0}, \qquad g_1|\mathbf{\Omega}_e|^2(\lambda^e \times \mu^e) = (\widetilde{A}_{11})_e(\lambda^e \times \mu^e),$$
$$(\widetilde{A}_{21})_e(\lambda^e \times \mu^e) = \mathbf{0}, \qquad g_2|\mathbf{\Omega}_e|^2(\lambda^e \times \mu^e) = (\widetilde{A}_{22})_e(\lambda^e \times \mu^e),$$

where $(\widetilde{A}_{ij})_e$ is the evaluation in the equilibria of $\widetilde{A}_{ij}$.

Concluding, we have the following relations:

$$(\widetilde{A}_{12})_e = 0, \qquad |\mathbf{\Omega}_e|^2 = \frac{(\widetilde{A}_{11})_e}{g_1} = \frac{(\widetilde{A}_{22})_e}{g_2}$$

and the proof is complete. □

**Proposition 8.** *If* $\mathbf{z}_e = \left(\mathbf{\Pi}_e, \lambda^e, \mathbf{p}_\lambda^e, \mu^e, \mathbf{p}_\mu^e\right)$ *is Lagrangian relative equilibria in an approximate dynamics of order k, then denoting by* $|\lambda^e| = Z$, $\left|\mu^e + \frac{m_2}{M_2}\lambda^e\right| = X$, $\left|\mu^e - \frac{m_1}{M_2}\lambda^e\right| = Y$, *the system of equations*

$$\begin{cases} X^{2k+3} = \sum_{i=0}^{k} \beta_i Z^3 X^{2(k-i)} \\ Y^{2k+3} = \sum_{i=0}^{k} \beta_i Z^3 Y^{2(k-i)} \end{cases} \tag{6}$$

*has positive real solutions.*

**Proof.** The proof follows by using the expressions of $\widetilde{A}_{ij}$ given in (A.3). □

**Remark 9.** The parameters that have influence in the study of the number of the different configurations of Lagrangian relative equilibria will be $Z$ and $\beta_i$ $(i = 1, 2, \ldots, k)$.

*5.2. Sufficient condition of existence*

If we fix $Z$ and there exist $X$ and $Y$ verifying the system of equations

$$\begin{cases} X^{2k+3} = \sum_{i=0}^{k} \beta_i Z^3 X^{2(k-i)} \\ Y^{2k+3} = \sum_{i=0}^{k} \beta_i Z^3 Y^{2(k-i)} \end{cases}$$

with respect to an appropriate reference system, we can build Lagrangian relative equilibria. If $X = Y \neq Z$ is a solution of the previous system, then $S_i$ $(i = 0, 1, 2)$ form an isosceles triangle. If $X \neq Y \neq Z$ then $S_i$ form a scalene triangle.

Proposition 10 shows the form of the Lagrangian relative equilibria when $S_0$, $S_1$, $S_2$ form an isosceles triangle. In a similar way proposition 11 describes the Lagrangian relative equilibria expressions when $S_0$, $S_1$, $S_2$ form a scalene triangle. Thus, we obtain the following results.

**Proposition 10.** *With respect to an appropriate reference system we have that* $\mathbf{z}_e = \left(\mathbf{\Pi}_e, \lambda^e, \mathbf{p}^e_\lambda, \mu^e, \mathbf{p}^e_\mu\right)$ *given by*

$$\begin{aligned} \lambda^e &= (x_1, y_1, 0), & \mathbf{p}^e_\lambda &= g_1\omega_e(-y_1, x_1, 0), \\ \mu^e &= (x_2, y_2, 0), & \mathbf{p}^e_\mu &= g_2\omega_e(-y_2, x_2, 0), \\ \mathbf{\Omega}_e &= (0, 0, \omega_e), & \mathbf{\Pi}_e &= (0, 0, C\omega_e + l), \end{aligned}$$

*with*

$$x_1 = Z, \qquad y_1 = 0, \qquad x_2 = \frac{Z(m_1 - m_2)}{2(m_1 + m_2)}, \qquad y_2 = \pm\frac{\sqrt{4X^2 - Z^2}}{2}$$

*and*

$$\omega_e^2 = \sum_{i=0}^{k} \frac{G(m_0 + m_1 + m_2)\beta_i}{X^{2i+3}}$$

*where* $\beta_0 = 1$, $\beta_i = \frac{\alpha_i}{m_0}$, *for* $i \geqslant 1$, *are isosceles Lagrangian relative equilibria. Moreover, the total angular momentum vector of the system is given by*

$$\mathbf{L} = \left(0, 0, C\omega_e + l + \omega_e^2 \sum_{i=1}^{2} g_i \left(x_i^2 + y_i^2\right)\right).$$

**Proposition 11.** *With respect to an appropriate reference system we have that* $\mathbf{z}_e = \left(\mathbf{\Pi}_e, \lambda^e, \mathbf{p}^e_\lambda, \mu^e, \mathbf{p}^e_\mu\right)$ *given by*

$$\begin{aligned} \lambda^e &= (x_1, y_1, 0), & \mathbf{p}^e_\lambda &= g_1\omega_e(-y_1, x_1, 0), \\ \mu^e &= (x_2, y_2, 0), & \mathbf{p}^e_\mu &= g_2\omega_e(-y_2, x_2, 0), \\ \mathbf{\Omega}_e &= (0, 0, \omega_e), & \mathbf{\Pi}_e &= (0, 0, C\omega_e + l), \end{aligned}$$

*with*

$$x_1 = Z, \qquad y_1 = 0$$

$$x_2 = \frac{m_1(X^2 + Z^2 - Y^2) - m_2(Y^2 + Z^2 - X^2)}{2(m_1 + m_2)Z}$$

$$y_2 = \pm\frac{\sqrt{(Z + X + Y)(Z + X - Y)(Z + Y - X)(X + Y - Z)}}{2Z}$$

*and*

$$\omega_e^2 = \sum_{i=0}^{k} \frac{Gm_1(m_0+m_1+m_2)\beta_i}{(m_1+m_2)X^{2i+3}} + \sum_{i=0}^{k} \frac{Gm_2(m_0+m_1+m_2)\beta_i}{(m_1+m_2)Y^{2i+3}}$$

*where* $\beta_0 = 1$, $\beta_i = \frac{\alpha_i}{m_0}$, *for* $i \geqslant 1$, *are scalene Lagrangian relative equilibria. Moreover, the total angular momentum vector of the system is given by*

$$\mathbf{L} = \left(0, 0, C\omega_e + l + \omega_e^2 \sum_{i=1}^{2} g_i \left(x_i^2 + y_i^2\right)\right).$$

In what follows we study the Lagrangian relative equilibria in the approximate dynamics of orders 0 and 1 respectively.

### *5.3. Lagrangian relative equilibria in an approximate dynamics of order zero*

When $k = 0$, equations (6) are

$$\begin{cases} X^3 = Z^3 \\ Y^3 = Z^3 \end{cases}$$

and then we easily deduce that $X = Y = Z$. This means that $S_0$, $S_1$ and $S_2$ form an equilateral triangle. Moreover,

$$|\boldsymbol{\Omega}_e|^2 = \frac{G(m_0+m_1+m_2)}{Z^3}.$$

On the other hand, a parametrization of $\mathbf{z}_e = (\boldsymbol{\Pi}_e, \lambda^e, \mathbf{p}_\lambda^e, \mu^e, \mathbf{p}_\mu^e)$ is given by

$$\begin{aligned} \lambda^e &= (x_1, y_1, 0), & \mathbf{p}_\lambda^e &= g_1\omega_e(-y_1, x_1, 0), \\ \mu^e &= (x_2, y_2, 0), & \mathbf{p}_\mu^e &= g_2\omega_e(-y_2, x_2, 0), \\ \boldsymbol{\Omega}_e &= (0, 0, \omega_e), & \boldsymbol{\Pi}_e &= (0, 0, C\omega_e + l), \end{aligned}$$

where

$$x_1 = Z, \qquad y_1 = 0, \qquad x_2 = \frac{Z(m_1-m_2)}{2(m_1+m_2)}, \qquad y_2 = \pm\frac{\sqrt{3}Z}{2}.$$

This parametrization of the relative equilibria will play a key role in the study of their stability properties.

### *5.4. Lagrangian relative equilibria in an approximate dynamics of order 1*

For $k = 1$, equations (6) are

$$\begin{cases} X^5 - Z^3X^2 - \beta_1 Z^3 = 0 \\ Y^5 - Z^3Y^2 - \beta_1 Z^3 = 0 \end{cases} \tag{7}$$

where $Z$ and $\beta_1$ are parameters. We study the number of positive real roots of the polynomial

$$p(X) = X^5 - Z^3X^2 - \beta_1 Z^3$$

according to the values of the parameters $Z$ and $\beta_1$.

Applying Descartes' rule of signs, if $\beta_1 \geqslant 0$, then this polynomial can only have a positive real root.

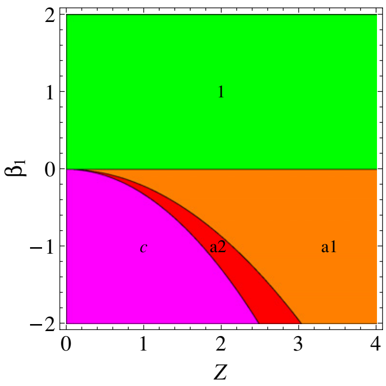


**Figure 2.** Bifurcations of the equilibria in the plane $\beta_1 Z$.

If $\beta_1 < 0$, then we can have two positive real roots, a real root (positive) or none. The discriminant of the polynomial, denoted by discrim$(p, X)$, is given by

$$\text{discrim}(p, X) = \beta_1 Z^{12}\left(3125\beta_1^3 + 108Z^6\right).$$

Then if discrim$(p, X) < 0$, the polynomial $p$ has two real roots, if discrim$(p, x) = 0$, it has a positive double root, while if discrim$(p, x) > 0$, it has no positive root.

The discriminant is zero when the following relation is verified:

$$\beta_1 = -\frac{3\sqrt[3]{20}}{25} Z^2.$$

By the previous results we can make a complete study of the bifurcations of the equilibria in an approximate dynamics of order 1, see figure 2.

**Proposition 12.** *Let* $\mathbf{z}_e = \left(\mathbf{\Pi}_e, \lambda^e, \mathbf{p}_\lambda^e, \mu^e, \mathbf{p}_\mu^e\right)$ *be a Lagrangian relative equilibria, in an approximate dynamics of order 1.*

*(1) If* $\beta_1 \geqslant 0$ *(gyrostat oblate), an only 2-parametric family exists forming* $S_0$, $S_1$, $S_2$ *an isosceles triangle.*

*(2) If* $\beta_1 < 0$ *(gyrostat prolate), then*

- *(a1) if* $\frac{-7Z^2}{32} < \beta_1 < 0$*, there are two types of relative equilibria:*
  - *one 2-parametric family of relative equilibria forming* $S_0$, $S_1$, $S_2$ *an isosceles triangle with* $X = Y \neq Z$
  - *two 2-parametric families of relative equilibria forming* $S_0$, $S_1$, $S_2$ *a scalene triangle with* $X \neq Y \neq Z$;
- *(a2) if* $-\frac{3\sqrt[3]{20}}{25}Z^2 < \beta_1 < \frac{-7Z^2}{32}$*, there are two types of relative equilibria:*
  - *two 2-parametric families of relative equilibria forming* $S_0$, $S_1$, $S_2$ *an isosceles triangle with* $X = Y \neq Z$
  - *four 2-parametric families of relative equilibria forming* $S_0$, $S_1$, $S_2$ *an scalene triangle with* $X \neq Y \neq Z$;
- *(b) if* $\beta_1 = -\frac{3\sqrt[3]{20}}{25}Z^2$*, an only 2-parametric family exists forming* $S_0$, $S_1$, $S_2$ *an isosceles triangle, with* $X = Y \neq Z$;
- *(c) if* $\beta_1 < -\frac{3\sqrt[3]{20}}{25}Z^2$*, relative equilibria do not exist.*

**Remark 13.** It is easy to see that when the gyrostat is oblate, in the previous equilibria, it rotates quicker around the principal axis of inertia $C$ than that when the gyrostat is prolate.

**Remark 14.** To study the Lagrangian relative equilibria in an approximate dynamics of order $k$, we should study the positive real solutions of the equation

$$X^{2k+3} - \sum_{i=0}^{k} \beta_i Z^3 X^{2(k-i)} = 0.$$

If we know the number of positive roots in the approximate dynamics of order $k$, we can know the number of positive roots of the polynomial equation that arises in the approximate dynamics of order $k + 1$. This study reduces to calculate the number of positive roots of the equation

$$\beta_{k+1} = \frac{X^2\left[X^{2k+3} - \sum_{i=0}^{k} \beta_i Z^3 X^{2(k-i)}\right]}{Z^3}.$$

### *5.5. Lagrangian relative equilibria of order 1 when $S_0$ has form close to a sphere*

If $S_0$ is close to a sphere, then $\beta_1 \approx 0$. To first order in $\beta_1$ the parametrization of $\mathbf{z}_e = (\mathbf{\Pi}_e, \lambda^e, \mathbf{p}_\lambda^e, \mu^e, \mathbf{p}_\mu^e)$ is given by

$$\lambda^e = (x_1, y_1, 0), \qquad \mathbf{p}_\lambda^e = g_1\omega_e(-y_1, x_1, 0),$$
$$\mu^e = (x_2, y_2, 0), \qquad \mathbf{p}_\mu^e = g_2\omega_e(-y_2, x_2, 0),$$
$$\mathbf{\Omega}_e = (0, 0, \omega_e), \qquad \mathbf{\Pi}_e = (0, 0, C\omega_e + l),$$

where

$$x_1 = Z, \qquad y_1 = 0, \qquad x_2 = \frac{Z(m_1 - m_2)}{2(m_1 + m_2)},$$

$$y_2 = \pm\left(\frac{\sqrt{3}Z}{2} + \frac{2\sqrt{3}}{9Z}\beta_1 + o(\beta_1)\right).$$

## 6. Linear stability of the Lagrangian relative equilibrium

The tangent flow of equations (3) at the equilibrium $\mathbf{z}_e$ is

$$\frac{\mathrm{d}\delta\mathbf{z}}{\mathrm{d}t} = \mathfrak{U}(\mathbf{z}_e)\delta\mathbf{z}$$

where $\delta\mathbf{z} = \mathbf{z} - \mathbf{z}_e$ and $\mathfrak{U}(\mathbf{z}_e)$ is the Jacobian matrix of (3) in $\mathbf{z}_e$.

### *6.1. Order zero approximate dynamics*

The characteristic polynomial of $\mathfrak{U}(\mathbf{z}_e)$ has the following expression:

$$P(\lambda) = \lambda^3(\lambda^2 + \Phi^2)\left(\lambda^2 + \omega_e^2\right)^3\left(\lambda^4 + \omega_e^2\lambda^2 + q\right) \tag{8}$$

where

$$\omega_e^2 = \frac{G(m_0 + m_1 + m_2)}{Z^3}, \qquad q = \frac{27G^2(m_1m_0 + m_2m_0 + m_1m_2)}{4Z^6}$$

and $\Phi = \frac{(C-A)\omega_e + l}{A}$.

Then the following results are verified.

**Proposition 15.** *$\mathbf{z}_e$ is spectral stable if*

$$(m_0 + m_2 + m_1)^2 \geqslant 27(m_1 m_0 + m_2 m_0 + m_1 m_2). \quad (9)$$

*If*

$$(m_0 + m_2 + m_1)^2 < 27(m_1 m_0 + m_2 m_0 + m_1 m_2),$$

*then $\mathbf{z}_e$ is unstable.*

**Proof.** The proof follows from the form of the minimum polynomial of $\mathfrak{U}(\mathbf{z}_e)$ which has the following expression:

$$Q(\lambda) = \lambda^2(\lambda^2 + \Phi^2)\left(\lambda^2 + \omega_e^2\right)\left(\lambda^4 + \omega_e^2\lambda^2 + q\right).$$

□

**Proposition 16.** *The linear system*

$$\frac{\mathrm{d}\delta\mathbf{z}}{\mathrm{d}t} = \mathfrak{U}(\mathbf{z}_e)\delta\mathbf{z}$$

*is unstable.*

**Proof.** In this case the minimum polynomial of $\mathfrak{U}(\mathbf{z}_e)$ has the zero as the double root; that is why the matrix $\mathfrak{U}(\mathbf{z}_e)$ is not diagonalizable, and the proof is complete. □

### *6.2. Order 1 approximate dynamics*

Similar results show that the characteristic polynomial in an order 1 approximate dynamics has the following expression:

$$P(\lambda) = \lambda(\lambda^2 + \Phi^2)(\lambda^2 + m)(\lambda^2 + n)h(\lambda)$$

with

$$h(\lambda) = \lambda^8 + p\lambda^6 + q\lambda^4 + r\lambda^2 + s.$$

Thus, we have the following result.

**Proposition 17.** *The Lagrangian relative equilibria in order 1 approximate dynamics are spectral stable (lineary stable) if the following conditions are verified:*

$$p^2q^2 - 3rp^3 - 6p^2s - 4q^3 + 14pqr + 16qs - 18r^2 \geqslant 0(> 0)$$
$$p^2qr - 48sr - 9sp^3 + 32pqs - 4q^2r + 3pr^2 \geqslant 0(> 0)$$
$$r, s \geqslant 0\,(> 0),\ 3p^2 - 8q \geqslant 0\,(> 0),\ pr - 16s \geqslant 0\,(> 0)$$
$$m, n \geqslant 0\,(> 0)$$
$$\mathrm{discrim}(h) \geqslant 0\,(> 0)$$

*where*

$$\begin{aligned}\mathrm{discrim}\,(h) = {} & 18p^3rqs - 4p^3r^3 - 128q^2s^2 + 16q^4s - 4q^3r^2 - 27p^4s^2 \\ & - 80prq^2s + 256s^3 - 27r^4 - 6p^2r^2s - 192prs^2 + 18pr^3q + 144qp^2s^2 \\ & + q^2p^2r^2 - 4q^3p^2s + 144sr^2q.\end{aligned}$$

**Proof.** The coefficients of the characteristic polynomial are expressed as a function of the parameters of our problem, i.e. the masses and the coefficient $\beta_1$, and the proof follows from the application of the Sturm theorem. □

**Remark 18.** If $\mathbf{z}_e$ is an arbitrary relative equilibrium, the conditions of the statement of proposition 17 have very complicated expressions in the parameters of the problem, and can only be studied via numerical analysis.

If $S_0$ is close to a sphere, the coefficients of $P$, to first order in the parameter $\beta_1$, hold the following relationships:

$$m = \frac{G(m_0+m_1+m_2)}{Z^3} + o_1(\beta_1), \qquad n = \frac{G(m_0+m_1+m_2)}{Z^3} + o_2(\beta_1),$$

$$s = o_3(\beta_1),$$

$$r = \frac{27G^3(m_0+m_1+m_2)(m_1m_0+m_2m_0+m_1m_2)}{4Z^9} + o_4(\beta_1),$$

$$q = \frac{G^2\left(4m_0^2+4m_1^2+4m_2^2+35m_0m_1+35m_0m_2+35m_1m_2\right)}{4Z^6} + o_5(\beta_1),$$

$$p = \frac{2G(m_0+m_1+m_2)}{Z^3} + o_6(\beta_1).$$

If the function

$$o_3(\beta_1) = \frac{81G^4m_0(m_1+m_2)(m_0+m_1+m_2)^2}{4}\beta_1 + o\left(\beta_1^2\right)$$

is positive and

$$(m_0+m_2+m_1)^2 > 27(m_1m_0+m_2m_0+m_1m_2), \tag{10}$$

then $\mathbf{z}_e$ is linearly stable in order 1 approximate dynamics. Then if $S_0$ is close to a sphere and $C > A$, then $\mathbf{z}_e$ is linearly stable if (10) is verified.

For $\beta_1 = \frac{3(C-A)}{2m_0} > 0$ in the domain of parameters $m_0, m_1, m_2$ the surface $(m_0+m_2+m_1)^2 = 27(m_1m_0+m_2m_0+m_1m_2)$ is a ruled hyperboloid. Figures 3 and 4 illustrate these facts.

## 7. Conclusions and future works

The approximate dynamics of a gyrostat (or rigid body) in Newtonian interaction with two spherical or punctual rigid bodies has been considered. For orders 0 and 1 of the approximate dynamics we provide a complete study of Lagrangian relative equilibria. The bifurcations of the Lagrangian relative equilibria are completely described for an approximate dynamics of order 1. Necessary and sufficient conditions are given for the linear stability of the Lagrangian relative equilibria in zero-order and one-order approximate dynamics if the gyrostat $S_0$ has form close to a sphere.

Several results obtained in previous works by classical methods have been obtained and generalized in a different way. Other results, not previously considered, have been studied. These developments help to understand real physical situations see figures 5–7.

The methods employed in this work are susceptible of being used in similar problems. Numerous problems are open, and among them it is necessary to consider the study of the 'inclined' relative equilibria, in which $\mathbf{\Omega}_e$ form an angle $\alpha \neq 0$ and $\pi/2$ with the vector $\lambda^e \times \mu^e$. The study of the nonlinear stability of the relative equilibria obtained here is the natural continuation of this work.

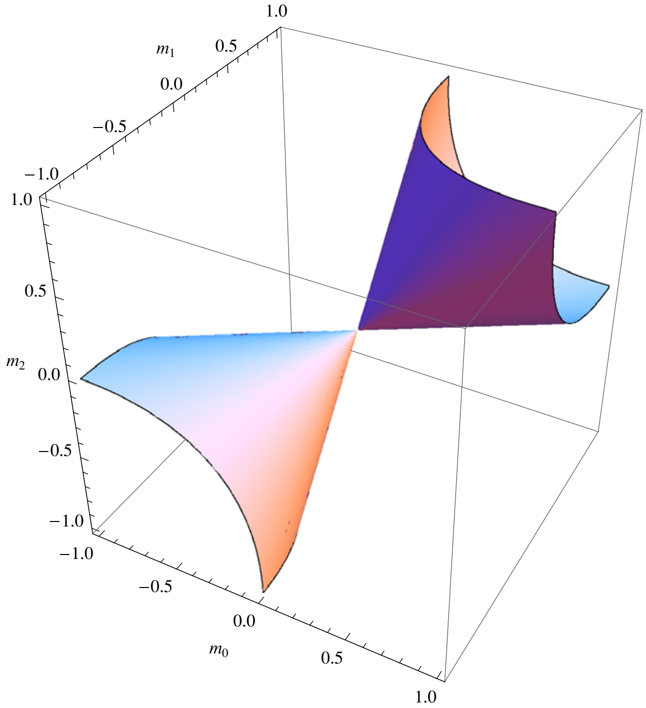


**Figure 3.** The ruled surface.

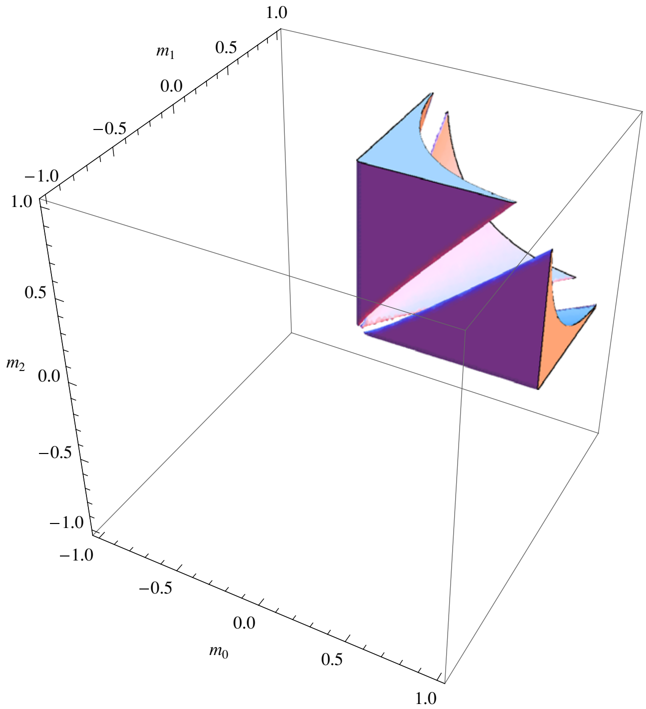


**Figure 4.** Region of linear stability.

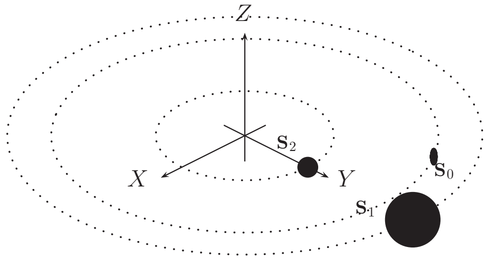


**Figure 5.** Isosceles Lagrangian equilibrium with $S_0$ a prolate gyrostat. This equilibrium is always unstable for all values of the parameters.

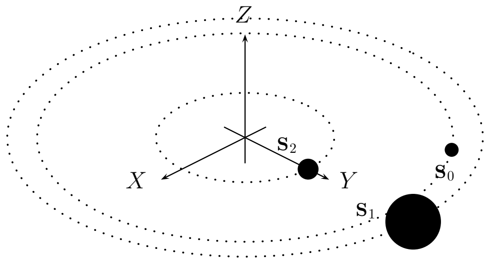


**Figure 6.** Equilateral Lagrangian equilibrium with $S_0$ a spherical gyrostat. This equilibrium is spectrally stable when $(m_0 + m_2 + m_1)^2 > 27(m_1m_0 + m_2m_0 + m_1m_2)$.

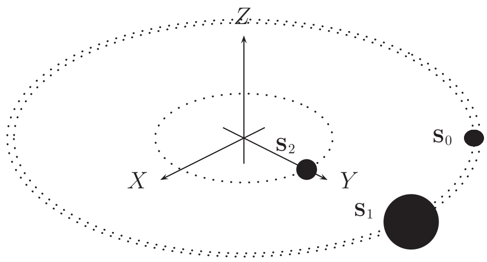


**Figure 7.** Isosceles Lagrangian equilibrium with $S_0$ is an oblate gyrostat. This equilibrium is linearly stable when $\beta_1$ is a very small positive value and $(m_0 + m_2 + m_1)^2 > 27(m_1m_0 + m_2m_0 + m_1m_2)$.

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## Appendix

The following expressions are obtained for the potential $\mathcal{V}^{(k)}$:

$$(\nabla_\lambda \mathcal{V}^{(k)})_e = \left( \frac{G m_1 m_2 \lambda^e}{|\lambda^e|^3} + \frac{G m_1 m_2}{M_2} \sum_{i=0}^{k} \frac{\alpha_i \left(\mu^e + \frac{m_1}{M_2}\lambda^e\right)}{|\mu^e + \frac{m_1}{M_2}\lambda^e|^{2i+3}} \right.$$
$$\left. - \frac{G m_1 m_2}{M_2} \sum_{i=0}^{k} \frac{\alpha_i \left(\mu^e - \frac{m_2}{M_2}\lambda^e\right)}{|\mu^e - \frac{m_2}{M_2}\lambda^e|^{2i+3}} \right)$$
$$(\nabla_\mu \mathcal{V}^{(k)})_e = G m_1 \sum_{i=0}^{k} \frac{\alpha_i \left(\mu^e + \frac{m_1}{M_2}\lambda^e\right)}{|\mu^e + \frac{m_1}{M_2}\lambda^e|^{2i+3}} + G m_2 \sum_{i=0}^{k} \frac{\alpha_i \left(\mu^e - \frac{m_2}{M_2}\lambda^e\right)}{|\mu^e - \frac{m_2}{M_2}\lambda^e|^{2i+3}}. \tag{A.1}$$

Also, the following identities are verified:

$$(\nabla_\lambda \mathcal{V}^{(k)})_e = (\widetilde{A}_{11})_e \lambda^e + (\widetilde{A}_{12})_e \mu^e, \qquad (\nabla_\mu \mathcal{V}^{(k)})_e = (\widetilde{A}_{21})_e \lambda^e + (\widetilde{A}_{22})_e \mu^e \tag{A.2}$$

where

$$\widetilde{A}_{11}(\lambda^e, \mu^e) = \frac{G m_1 m_2}{|\lambda|^3} + \frac{G m_1 m_2^2}{M_2^2} \left( \sum_{i=0}^{k} \frac{\alpha_i}{|\mu^e + \frac{m_1}{M_2}\lambda^e|^{2i+3}} \right)$$
$$+ \frac{G m_1^2 m_2}{M_2^2} \left( \sum_{i=0}^{k} \frac{\alpha_i}{|\mu^e - \frac{m_2}{M_2}\lambda^e|^{2i+3}} \right)$$
$$\widetilde{A}_{12}(\lambda^e, \mu^e) = \frac{G m_1 m_2}{M_2} \left( \sum_{i=0}^{k} \frac{\alpha_i}{|\mu^e + \frac{m_1}{M_2}\lambda^e|^{2i+3}} - \sum_{i=0}^{k} \frac{\alpha_i}{|\mu^e - \frac{m_2}{M_2}\lambda^e|^{2i+3}} \right)$$
$$\widetilde{A}_{21}(\lambda^e, \mu^e) = \widetilde{A}_{12}(\lambda^e, \mu^e)$$
$$\widetilde{A}_{22}(\lambda^e, \mu^e) = G m_1 \left( \sum_{i=0}^{k} \frac{\alpha_i}{|\mu^e + \frac{m_1}{M_2}\lambda^e|^{2i+3}} \right) + G m_2 \left( \sum_{i=0}^{k} \frac{\alpha_i}{|\mu^e - \frac{m_2}{M_2}\lambda^e|^{2i+3}} \right) \tag{A.3}$$

with $\alpha_0 = m_0$, $\alpha_1 = 3(C - A)/2$ and $\alpha_i = (2i + 1)A_{2i}$ for $i \geqslant 2$.

## References

[1] Duboshin G N 1984 The problem of three rigid bodies *Celest. Mech. Dyn. Astron.* **33** 31–47
[2] Guirao J L G and Vera J A 2010 Dynamics of a gyrostat on cylindrical and inclined Eulerian equilibria in the three body problem *Acta Astronaut.* **66** 595–604
[3] Leimanis E 1965 *The General Problem of the Motion of Coupled Rigid Bodies About a Fixed Point* (Berlin: Springer)
[4] Maciejewski A 1995 Reduction, relative equilibria and potential in the two rigid bodies problem *Celest. Mech. Dyn. Astron.* **63** 1–28
[5] Mondéjar F and Vigueras A 1999 The Hamiltonian dynamics of the two gyrostats problem *Celest. Mech. Dyn. Astron.* **73** 303–12
[6] Vidiakin V V 1977 Euler solutions in the problem of translational–rotational motion of three-rigid bodies *Celest. Mech. Dyn. Astron.* **16** 509–26
[7] Vera J A 2004 Reducciones, equilibrios y estabilidad en dinámica de sólidos rígidos y giróstatos *PhD Dissertation* Universidad Politécnica de Cartagena, Spain

[8] Vera J A and Vigueras A 2004 Reduction, relative equilibria and stability for a gyrostat in the *n*-body problem *Monografías del Seminario Matemático García de Galdeano (VIII Journées Zaragoza-Pau de Mathematiques Appliquées et de Statistiques)* vol 31 ed M C López de Silanes *et al* (Zaragoza, Spain: Servicio de Publicaciones de la Universidad de Zaragoza) pp 257–71

[9] Vera J A and Vigueras A 2006 Hamiltonian dynamics of a gyrostat in the *n*-body problem: relative equilibria *Celest. Mech. Dyn. Astron.* **94** 289–315

[10] Wang L S, Krishnaprasad P S and Maddocks J H 1991 Hamiltonian dynamics of a rigid body in a central gravitational field *Celest. Mech. Dyn. Astron.* **50** 349–86